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HIGHER-ORDER SENSITIVITY TO IMPERFECTIONS IN BIFURCATION BUCKLING ANALYSIS

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Abstract—An alternative to the usual procedure of determining the maximum loads and corresponding coordinates as expansions in terms of an imperfection parameter is presented. The present approach is direct and does not require the development of inverse expansions. The procedure is based on the concept of least-degeneracy, which has also been used as the basis for the analysis in a wide variety of other applications. Use of the method is demonstrated by applying it to two examples: one symmetric bifurcation and one asymmetric bifurcation. It is shown that the present method very efficiently reproduces known results. A brief discussion of the consistent retention of terms in the various expansions is also presented.

1. INTRODUCTION

The general theory of elastic stability was originally formulated for continuous systems by Koiter (1945), and a number of researchers have contributed to a similar development for discrete systems. An account of the achievements in the theory in terms of generalized coordinates may be found in the books by Croll and Walker (1972), Thompson and Hunt (1973), Huseyin (1975) and, more recently, El Naschie (1990). Distinct critical states in the theory are found to be of two kinds: limit points and bifurcations points, and special attention is given to the sensitivity of critical states with respect to structural imperfections. An account of imperfection sensitivity may be found in the literature mentioned above and in the many references cited therein.

For critical states that are characterized as bifurcations, the usual approach to imperfection sensitivity is to define a new parameter ε, which serves as a measure of the amplitude of a given imperfection, and write the total potential energy V of the system in terms of the usual control λ and response Q_j parameters plus the new imperfection parameter. A critical state satisfies two conditions: equilibrium (associated with the zero of the first variation of V) and critical stability (associated with the zero of the second variation of V) (Croll and Walker, 1972; Thompson and Hunt, 1973; Huseyin, 1975; El Naschie, 1990). Imperfection sensitivity is obtained by perturbing these two equations in order to find the critical state as a function of the imperfection amplitude. This dependence is usually written as $\lambda_c = \lambda_c(\varepsilon)$. The problem is that in bifurcation analysis such a relation has an infinite slope at $\varepsilon = 0$ and the expansion cannot be made from that point by the use of regular perturbation methods.

One solution to this problem is described by Thompson and Hunt (1973). The perturbation is carried out with respect to another parameter, say Q_1, and series are found in the form $\lambda_c = \lambda_c(Q_1)$; $\varepsilon = \varepsilon(Q_1)$; $Q_{jc} = Q_{jc}(Q_1)$, where c denotes critical values and $j > 1$. Then the middle expansion is inverted to provide $Q_{1c} = Q_{1c}(\varepsilon)$, and the result is substituted

into the two remaining series to produce the desired expansions. The three primary expansions are regular and can be obtained in the usual straightforward way. The final expansions are in terms of fractional powers of ε and have infinite slopes at $\varepsilon = 0$.

In this paper, we describe an alternative to this procedure, which does not require intermediate expansions. The relationship between λ_c and ε as well as those between the Q_{jc} and ε are determined directly. The straightforward procedure of assuming expansions of the zeros in integral powers of ε does not produce results because the zeros of the first and second variations of the potential energy when $\varepsilon = 0$ are always repeated roots. A consequence of repeated roots is that the expansions do not go in integral powers of ε. Indeed the powers of ε cannot be assumed *a priori* and must be determined as part of the solution. Similar problems arise in the analysis of dynamic systems when two or more equilibrium points coalesce. Moreover, in analysing boundary-layer phenomena, whether in plates, electrical conductors, or high Reynolds number flows, one must "stretch" one or more coordinates. Typically, the stretching goes according to a nonintegral power of a naturally occurring, small parameter (such as the square root of the reciprocal of the Reynolds number in the case of flows). Determining the proper stretching in these singular-perturbation problems is a key element of the analysis, and that procedure is quite similar to the one employed here. The problem is discussed in the contexts of single algebraic equations and boundary layers in the text by Nayfeh (1981).

2. A SUMMARY OF THE AVAILABLE RESULTS

We shall follow the perturbation of Thompson and Hunt (1972), because it develops the formulation not only for first, but also for higher order terms in the imperfection-sensitivity perturbation analysis. The total potential energy of the system is given by $V = V(Q_j, \lambda)$; the conditions of equilibrium and critical stability are written as

$$V_{,i} = \frac{\partial V}{\partial Q_i} = 0 \tag{1}$$

$$V_{,ij}x_j = \frac{\partial^2 V}{\partial Q_i \partial Q_j} x_j = 0, \tag{2}$$

where eqn (2) is an eigenvalue problem in terms of the eigenvalue λ and the eigenvector x_j. Typically, eqns (1) and (2) have multiple roots; this is demonstrated in the section on applications below.

Following the general theory of elastic stability, one studies the critical state for $\varepsilon = 0$ by writing the perturbation expansions

$$\lambda(s) = \lambda_c + \lambda^{(1)}s + \frac{1}{2}\lambda^{(2)}s^2 + \cdots \tag{3}$$

$$Q_j(s) = Q_j^c + Q_j^{(1)}s + \frac{1}{2}Q_j^{(2)}s^2 + \cdots, \tag{4}$$

where s is a suitable perturbation parameter (say $s = Q_1$) and the coefficients are the derivatives

$$(\ )^{(n)} \equiv \left.\frac{\partial^n(\ )}{\partial s^n}\right|_c$$

evaluated at the critical state.

For the imperfection-sensitivity analysis, we include the new parameter ε (a measure of the amplitude of the imperfection considered), so that $V = V(Q_j, \lambda; \varepsilon)$. With reference to

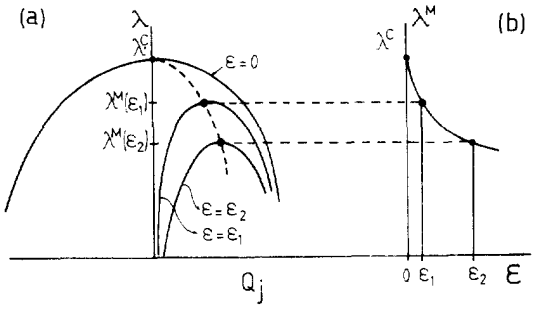

Fig. 1. (a) Equilibrium paths for different imperfection amplitudes. (b) Imperfection-sensitivity of the maximum load, $\lambda^M = \lambda_{max}$.

Fig. 1, eqns (1) and (2) represent the conditions of the maximum in each curve, for a specific imperfection. The particular example presented in Fig. 1 is an unstable symmetric bifurcation when $\varepsilon = 0$, but the imperfection breaks the bifurcation and leads to a nonlinear path with a maximum $\lambda_{max}(\varepsilon)$. At $\varepsilon = 0$ in Fig. 1b, the curve is tangent to the λ axis; this means that a regular perturbation analysis cannot be carried out to represent the curve. Hunt (1971) proposed an alternative perturbation scheme by choosing an intermediate variable as the perturbation parameter, namely $s = Q_1$. Thus, the following equations are obtained:

$$\lambda_{max} = \lambda_c + \lambda_1 Q_1 + \frac{1}{2}\lambda_2 Q_1^2 + \cdots \tag{5}$$

$$Q_{jmax} = Q_{jc} + Q_{j1} Q_1 + \frac{1}{2} Q_{j2} Q_1^2 + \cdots \tag{6}$$

$$\varepsilon = \varepsilon_0 + \varepsilon_1 Q_1 + \frac{1}{2}\varepsilon_2 Q_1^2 + \frac{1}{3!}\varepsilon_3 Q_1^3, \tag{7}$$

where λ_1, Q_{j1} and ε_1 are the first derivatives with respect to Q_1; and λ_2, Q_{j2} and ε_2 are the second derivatives. The difference between eqns (5)–(7) and (3) and (4) is that the latter are computed by perturbing eqn (1) for $\varepsilon = 0$; while the former are calculated from perturbations of (1) and (2) for $\varepsilon \neq 0$. Once the coefficients of eqns (5)–(7) are calculated, the series are inverted to obtain $\lambda_{max} = \lambda(\varepsilon)$. The explicit form for these series, for the case of asymmetric bifurcation, is

$$\lambda = \lambda_c + \alpha(\varepsilon)^{1/2} + \beta(\varepsilon) + \gamma(\varepsilon)^{3/2} + \cdots, \tag{8}$$

where

$$\alpha = \pm\left(\frac{2}{\varepsilon_2}\right)^{1/2} \lambda_1 \tag{9a}$$

$$\beta = \frac{\lambda_2}{\varepsilon_2} - \frac{\lambda_1 \varepsilon_3}{3(\varepsilon_2)^2} \tag{9b}$$

$$\gamma = \pm\left(\frac{2}{\varepsilon_2}\right)^{1/2} \left\{\frac{\lambda_3}{3\lambda_2} - \frac{\lambda_2 \varepsilon_3}{3(\varepsilon_2)^2} + \frac{\lambda_1}{36(\varepsilon_2)^3}[5(\varepsilon_3)^2 - 3\varepsilon_2 \varepsilon_4]\right\}. \tag{9c}$$

In eqns (8) and (9), the notation $1\ 2-$ means that only the positive square root is considered, and the sign of γ depends upon the sign of α. For the symmetric bifurcation the sensitivity equation is

$$\lambda_{\max} = \lambda_c + \alpha\varepsilon^{2/3} + \beta\varepsilon + \gamma\varepsilon^{4/3} + \cdots, \tag{10}$$

where

$$\alpha = \frac{1}{2}\lambda_2\left(\frac{6}{\varepsilon_2}\right)^{2/3} \tag{11a}$$

$$\beta = \frac{1}{2}\left[\frac{2\lambda_3}{\varepsilon_3} - \frac{\lambda_2\varepsilon_4}{\varepsilon_3}\right] \tag{11b}$$

$$\gamma = \left(\frac{6}{\varepsilon_3}\right)^{1/3}\left[\frac{\lambda_4}{4\varepsilon_3} - \frac{\lambda_3\varepsilon_4}{4(\varepsilon_3)^2} - \frac{\lambda_2\varepsilon_5}{10(\varepsilon_3)^2} + \frac{7}{48}\frac{\lambda_2(\varepsilon_4)^2}{(\varepsilon_3)^3}\right]. \tag{11c}$$

3. PROPOSED SOLUTION

Rather than using an intermediate perturbation parameter, as in eqns (5)–(7), we employ $s = \varepsilon$. We assume that the expansions for the maximum load, $\lambda_{\max}$, and the corresponding coordinate, $Q_{\max}$, have the following forms

$$\lambda_{\max} = \lambda_c + \lambda_1\varepsilon^M + \lambda_2\varepsilon^R + \lambda_3\varepsilon^T + \cdots \tag{12a}$$

$$Q_{j\max} = Q_{jc} + Q_{j1}\varepsilon^N + Q_{j2}\varepsilon^P + Q_{j3}\varepsilon^S + \cdots, \tag{12b}$$

where $j = 1, 2, \cdots K$; and the positive exponents M, N, P, R, S, and T as well as the coefficients λ_c, λ_1, λ_2, $\cdots$ and Q_{jc}, Q_{j1}, $\cdots$ are unknown. K is the order (i.e. number of coordinates needed to describe the state) of the system. We note that the coefficients λ_1, λ_2, $\cdots$, Q_{j1}, Q_{j2}, $\cdots$ in eqns (12a, b) are not related to those in eqns (5)–(7).

First, we substitute eqns (12a, b) into eqns (1) and (2); then we set $\varepsilon = 0$ and solve for λ_c and Q_{jc}. This is precisely the stability problem for no imperfections. Next, we include λ_1 and the Q_{j1} as well as λ_c and the Q_{jc}, but consider the remaining λ_n and Q_{jn} to be zero; then we select the heretofore arbitrary exponents M and N so as to obtain the "least-degenerate" two-term approximation (we shall explain and illustrate the concept of least degeneracy in the examples that follow); finally, we solve for λ_1 and the Q_{j1}. Next we add λ_2 and the Q_{j2} to the expansion; and repeat the procedure. Any desired number of terms may be included.

The proposed technique is applied in the next two sections to problems of symmetric and asymmetric bifurcation with one degree of freedom, in order to highlight the main features of the procedure.

4. SYMMETRIC BIFURCATION

The first example is a one-degree-of-freedom problem discussed by Croll and Walker (1972: Chap. 5) and illustrated in Fig. 2. The exact form of the total potential energy is

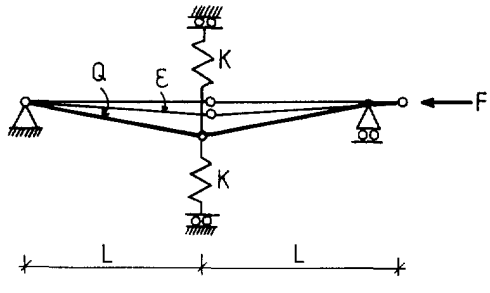

Fig. 2. Example considered for symmetric bifurcation.

$$V(Q,\lambda;\varepsilon) = (\sin Q - \sin\varepsilon)^2 - 2\lambda(\cos\varepsilon - \cos Q), \tag{13}$$

where $\lambda \equiv F/KL$, and Q and ε have the meaning of angles, as shown in Fig. 2. Here we do not need a subscript on Q.

We begin by expanding the trigonometric functions as follows:

$$\sin Q \doteq Q - \frac{1}{6}Q^3 + \frac{1}{120}Q^5 \tag{14a}$$

$$\cos Q \doteq 1 - \frac{1}{2}Q^2 + \frac{1}{24}Q^4 - \frac{1}{720}Q^6 \tag{14b}$$

$$\sin\varepsilon \doteq \varepsilon \tag{14c}$$

$$\cos\varepsilon \doteq 1 \tag{14d}$$

Next we substitute eqns (14a–d) into (13), retain terms up to and including fourth order, and obtain the following approximate expression for V:

$$V(Q,\lambda;\varepsilon) = Q^2 - \frac{1}{3}Q^4 - \lambda\left(Q^2 - \frac{1}{12}Q^4\right) - \varepsilon\left(2Q - \frac{1}{3}Q^3\right). \tag{15a}$$

The equilibrium condition is given by

$$\frac{1}{2}\frac{dV}{dQ} = Q_{max} - \frac{2}{3}Q_{max}^3 - \lambda_{max}\left(Q_{max} - \frac{1}{6}Q_{max}^3\right) - \varepsilon\left(1 - \frac{1}{2}Q_{max}^2\right) = 0 \tag{15b}$$

and the condition of critical stability is given by

$$\frac{1}{2}\frac{d^2V}{dQ^2} = 1 - 2Q_{max}^2 - \lambda_{max}\left(1 - \frac{1}{2}Q_{max}^2\right) - \varepsilon Q_{max} = 0. \tag{15c}$$

When $\varepsilon = 0$, it follows from eqns (12a, b) that $\lambda_{max} = \lambda_c$ and $Q_{max} = Q_c$; then eqns (15b, c) reduce to

$$6(1-\lambda_c)Q_c - (4-\lambda_c)Q_c^3 = 0 \tag{16a}$$

and

$$2(1-\lambda_c) - (4-\lambda_c)Q_c^2 = 0. \tag{16b}$$

The roots are $\lambda_c = 1$ and $Q_c = 0$. We note the repeated roots.

To determine the influence of the imperfection ε, we begin by including the second terms in eqns (12a, b)

$$\lambda_{max} = 1 + \lambda_1\varepsilon^M \tag{17a}$$

$$Q_{max} = Q_1\varepsilon^N, \tag{17b}$$

where M and N are arbitrary, but greater than zero, and λ_1 and Q_1 are unknown. Substituting eqns (17a, b) into (15b, c), we obtain

$$\frac{1}{2}\frac{dV}{dQ} = Q_1^2\varepsilon^{2M+1} - Q_1^3\varepsilon^{3M} - 2\varepsilon - \frac{1}{12}Q_1^4\varepsilon^{4M+1} - 2\lambda_1 Q_1\varepsilon^{M+N} + \frac{1}{3}\lambda_1 Q_1^3\varepsilon^{3M+N} = 0 \quad (18a)$$

$$\frac{1}{2}\frac{d^2V}{dQ^2} = 2Q_1\varepsilon^{M+1} - 3Q_1^2\varepsilon^{2M} - \frac{1}{3}Q_1^3\varepsilon^{3M+1} - 2\lambda_1\varepsilon^N + \lambda_1 Q_1^2\varepsilon^{2M+N} = 0. \quad (18b)$$

Some reduction in the number of terms can be made immediately: for small ε and positive M and N, $\varepsilon^{4M+1} \ll \varepsilon^{2M+1}$, $\varepsilon^{3M+N} \ll \varepsilon^{3M}$, $\varepsilon^{2M+N} \ll \varepsilon^{2M}$, and $\varepsilon^{3M+1} \ll \varepsilon^{M+1}$ regardless of the choice of M and N, and the corresponding terms can be ignored in eqns (18a, b). The result is

$$Q_1^2\varepsilon^{2M+1} - Q_1^3\varepsilon^{3M} - 2\varepsilon - 2\lambda_1 Q_1\varepsilon^{M+N} = 0 \quad (18c)$$

$$2Q_1\varepsilon^{M+1} - 3Q_1^2\varepsilon^{2M} - 2\lambda_1\varepsilon^N = 0. \quad (18d)$$

To obtain the least-degenerate approximation from eqns (18c, d), we select M and N so that these equations retain the maximum number of terms and, hence, produce the maximum amount of information. Ideally, if it were possible, we would choose M and N such that

$$2M+1 = 3M = 1 = M+N. \quad (19a)$$

and

$$M+1 = 2M = N. \quad (19b)$$

Generally, this is not possible and the proper selection of M and N will only satisfy some of the relationships given in eqns (19a, b). The idea is to choose the "least-degenerate" combination. Next we consider the various possibilities one by one.

If we choose $2M+1 = 3M$ in eqn (19a), then $M = 1$, and it follows from eqn (19b) that $N = 2$. There is a single dominant term in eqn (18c), and it is -2ε, which leads to the inconsistent conclusion $2 = 0$ if $\varepsilon \neq 0$. Hence, this is not the proper choice for M and N.

If we choose $2M+1 = 1$, then $M = N = 0$. This choice will not produce any new information. It places Q_1 and λ_1 on the same level with the solution corresponding to $\varepsilon = 0$; hence, Q_1 and λ_1 are zero. This also is not the proper choice for M and N.

If we choose $3M = 1$, then $M = 1/3$ and $N = 2/3$. Equations (18c, d) reduce to

$$\varepsilon(-Q_1^3 - 2 - 2\lambda_1 Q_1) + O(\varepsilon^{5/3}) = 0 \quad (20a)$$

and

$$\varepsilon^{2/3}(-3Q_1^2 - 2\lambda_1) + O(\varepsilon^{4/3}) = 0. \quad (20b)$$

From which we find

$$Q_1 = 1 \quad (21a)$$

and

$$\lambda_1 = -3/2. \quad (21b)$$

This is the only choice for M and N that does not lead to inconsistencies and produces new information. Equations (20a, b) are called the least-degenerate form of eqns (18c, d), and the results in eqns (21a, b) are called the least-degenerate approximation. So far the expansions have the following forms:

$$Q_{\max} = \varepsilon^{1/3} \tag{22a}$$

and

$$\lambda_{\max} = 1 - \frac{3}{2}\varepsilon^{2/3}. \tag{22b}$$

These results agree with eqn (10).

To continue the process, we now add more terms to the expansions in eqns (12a, b):

$$Q_* = \varepsilon^{1/3} + Q_2\varepsilon^P \tag{23a}$$

$$\lambda_* = 1 - \frac{3}{2}\varepsilon^{2/3} + \lambda_2\varepsilon^R. \tag{23b}$$

We can follow the procedure outlined above to determine P and R, or we can be guided by eqns (20) and (22) and choose $P = 1$ and $R = 4/3$. (One can readily verify that the other choices for P and R lead to inconsistencies.) Then substituting eqns (23a, b) into eqns (15b, c) leads to

$$\varepsilon^{2/3}(1 - 4\lambda_2) + O(\varepsilon^{4/3}) = 0 \tag{24a}$$

$$\varepsilon^{4/3}(1 - 4\lambda_2 - 12Q_2) - O(\varepsilon^2) = 0. \tag{24b}$$

From which we find that

$$Q_2 = 0 \tag{25a}$$

and

$$\lambda_2 = \frac{1}{4} \tag{25b}$$

and that so far the expansions have the following forms:

$$Q_{\max} = \varepsilon^{1/3} + O(\varepsilon^{5/3}) \tag{26a}$$

$$\lambda_{\max} = 1 - \frac{3}{2}\varepsilon^{2/3} + \frac{1}{4}\varepsilon^{4/3} + O(\varepsilon^2). \tag{26b}$$

The order of the truncation error in eqn (26b) is ε^2; thus, if more terms in the expansions for $Q_{\max}$ and $\lambda_{\max}$ are to be found, then to be consistent we must include the second term in the expansion of $\cos\varepsilon$ given in eqn (14d). A related question is how many terms in the expansions of $Q_{\max}$ and $\lambda_{\max}$ are consistent with the fourth-order expansion of V given in eqn (15a). To answer this question, we include more terms in the expansion of V and write

$$V(Q, \lambda, \varepsilon) = Q^2 - \frac{1}{3}Q^4 + \frac{2}{45}Q^6 - \lambda\left(Q^2 - \frac{1}{12}Q^4 + \frac{1}{360}Q^6\right) - \varepsilon\left(2Q - \frac{1}{3}Q^3 + \frac{1}{60}Q^5\right) + \cdots \tag{27}$$

When we repeat the procedure described above, we find that when we include only fifth order terms in the expression for V the results are the same as those in eqns (26). But when sixth (and higher) order terms are included, we obtain

Table 1. Maximum load, λ_{max}, for different values of the imperfection parameter, ε, for symmetric bifuration

ε, λ_{max}	First order Eqn (20)	Second order	
		Eqn (24)	Eqn (30)
0.001	0.985		
0.005	0.956		
0.010	0.930		
0.050	0.796	0.800	0.803
0.100	0.677	0.689	0.694
0.200	0.487	0.516	0.531
0.300	0.328	0.378	0.403
0.400	0.186	0.259	0.296

$$Q_{max} = \varepsilon^{1/3} + \frac{1}{6}\varepsilon + O(\varepsilon^{5/3}) \tag{28a}$$

$$\lambda_{max} = 1 - \frac{3}{2}\varepsilon^{3/2} + \frac{3}{8}\varepsilon^{4/3} + O(\varepsilon^2). \tag{28b}$$

Comparing eqns (26) and (28), we find the last terms in the two sets of expansions are different; those in eqns (28a, b) are correct. Hence, if a three-term expansion for λ_{max} is desired, then V must contain sixth order terms.

For the present example, the results of first and second order perturbation analyses are compared in Table 1. As expected, the differences between first-order and second-order solutions in $\lambda_{max}(\varepsilon)$ grow with ε. For example, for a value $\varepsilon = 0.2$, a first order solution would predict $\lambda_{max} = 0.487$, while a second order solution computed from a quartic functional leads to 0.516. If the approximations in the function V are improved so that it includes terms with Q to the sixth power, the value of $\lambda_{max} = 0.531$. The quadratic approximation is conservative in the sense that it predicts values that are lower than the correct ones. In this case the first order perturbation solution is conservative with respect to the second order one.

5. ASYMMETRIC BIFURCATION

As a second example, we consider the one-degree-of-freedom model studied in Croll and Walker (1972: Chap. 6) and shown in Fig. 3. The exact expression for the potential energy in this case is given by

$$V(\Delta_1, \Delta_2, F; \varepsilon) = \frac{1}{2}K\Delta_2^2 - F\Delta_1, \tag{29a}$$

where

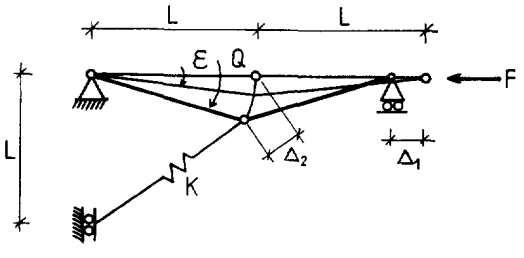

Fig. 3. Example considered for asymmetric bifurcation.

$$\Delta_1 = 2L(\cos\varepsilon - \cos Q) \tag{29b}$$

$$\Delta_2 = \sqrt{2}L[(1-\sin\varepsilon)^{1/2} - (1-\sin Q)^{1/2}] \tag{29c}$$

$$F = \frac{1}{4}KL\lambda. \tag{29d}$$

Next we substitute the following expansions

$$\begin{aligned}(1-\sin Q)^{1/2} &\doteq 1 - \frac{1}{2}Q - \frac{1}{8}Q^2 + \frac{1}{48}Q^3 + \frac{1}{384}Q^4 - \frac{1}{3840}Q^5\\ \cos Q &\doteq 1 - \frac{1}{2}Q^2 + \frac{1}{24}Q^4\\ (1-\sin\varepsilon)^{1/2} &\doteq 1 - \frac{1}{2}\varepsilon\\ \cos\varepsilon &\doteq 1\end{aligned} \tag{30}$$

into eqns (29) and obtain the following approximate expression for V:

$$V(Q_1,\lambda;\varepsilon) = Q^2 + \frac{1}{2}Q^3 - \frac{1}{48}Q^4 - \lambda\left(Q^2 - \frac{1}{12}Q^4\right) - \varepsilon\left(2Q + \frac{1}{2}Q^2 - \frac{1}{12}Q^3 - \frac{1}{96}Q^4\right). \tag{31a}$$

The equilibrium condition is given by

$$\frac{1}{2}\frac{dV}{dQ} = Q_{max} + \frac{3}{4}Q_{max}^2 - \frac{1}{24}Q_{max}^3 + \lambda_{max}\left(Q_{max} - \frac{1}{6}Q_{max}^3\right) - \varepsilon\left(1 + \frac{1}{2}Q_{max} - \frac{1}{8}Q_{max}^2 - \frac{1}{48}Q_{max}^3\right) = 0 \tag{31b}$$

and condition of critical stability is given by

$$\frac{1}{2}\frac{d^2V}{dQ^2} = 1 + \frac{3}{2}Q_{max} - \frac{1}{8}Q_{max}^2 - \lambda_{max}\left(1 - \frac{1}{2}Q_{max}^2\right) - \varepsilon\left(\frac{1}{2} - \frac{1}{4}Q_{max} - \frac{1}{16}Q_{max}^2\right) = 0. \tag{31c}$$

Next we substitute eqns (12a, b) into eqns (31b, c), set $\varepsilon = 0$, and solve for λ_c and Q_c. The solution is $\lambda_c = 1$ and $Q_c = 0$. To determine the influence of the imperfection, we substitute λ_c and Q_c into eqns (12a, b), retain the first two terms, substitute the result into eqns (31b, c), and obtain

$$\frac{1}{2}\frac{dV}{dQ} = -\varepsilon - \frac{1}{2}Q_1\varepsilon^{M+1} + \frac{3}{4}Q_1^2\varepsilon^{2M} - \lambda_1 Q_1\varepsilon^{M+N} = 0 \tag{32a}$$

$$\frac{1}{2}\frac{d^2V}{dQ^2} = -\varepsilon + 3Q_1\varepsilon^M - 2\lambda_1\varepsilon^N = 0. \tag{32b}$$

Next, we choose the values M, N, λ_1 and Q_1 that produce the least-degenerate approximation. Following the procedure described in the previous section, we find that $M = N = 1/2$, $Q_1^2 = -4/3$, and $\lambda = 3\sqrt{2}Q_1$. Thus, so far we have

$$\lambda_{max} = 1 \pm \sqrt{-3\varepsilon} \tag{33a}$$

$$Q_{max} = \pm \frac{2}{3}\sqrt{-3\varepsilon}. \tag{33b}$$

This is the same result obtained by Croll and Walker (1972). For values of $\varepsilon > 0$, λ_c and Q_c are imaginary; hence, the system is only sensitive to imperfections that correspond to negative values of ε. The result is known as the "one-half-power law".

To extend the expansions, we write

$$\lambda_{max} = 1 \pm \sqrt{-3\varepsilon + \lambda_2 \varepsilon^R} \tag{34a}$$

$$Q_{max} = \pm \frac{2}{3}\sqrt{-3\varepsilon + Q_2 \varepsilon^P}. \tag{34b}$$

Then we substitute eqns (34a, b) into eqns (31b, c) and find that $P = R = 1$, $\lambda_2 = -2/3$, and $Q_2 = 2(1 + \lambda_2)/3$. Hence, so far we have

$$\lambda_{max} = 1 + \sqrt{-\varepsilon' - \frac{2}{9}(\varepsilon')} \tag{35a}$$

$$Q_{max} = \frac{2}{3}\sqrt{-\varepsilon' + \frac{2}{27}(\varepsilon')}, \tag{35b}$$

where $\varepsilon' = 3\varepsilon$ and only the positive root is considered.

6. DISCUSSION AND CONCLUSIONS

We have described a new, direct procedure for obtaining the relationships among the critical loads, the critical coordinates, and a measure of the amplitude of the imperfection. Difficulties arise in obtaining these expansions because the roots of the conditions for equilibrium and critical stability are repeated in the absence of an imperfection. Thus, the expansions of the critical loads and coordinates do not go in integral powers of the imperfection describing parameter, and regular perturbation procedures are not applicable. Instead, we develop a direct approach that is similar to the one used in the analysis of singular perturbation problems to obtain the correct coordinate stretching. In the present approach, we assume the expansions for the critical loads and corresponding coordinates in terms of arbitrary powers of the imperfection describing parameter. Then, by using the principle of least-degeneracy, we determine the coefficients and exponents in the various terms.

Here we demonstrated the procedure by applying it for the first time to two well-known problems, each having a single degree of freedom. In future work, the procedure will be extended to systems having many degrees of freedom.

REFERENCES

Croll, J. G. A. and Walker, A. C. (1972). *Elements of Structural Stability*, Macmillan, London.
El Naschie, M. S. (1990). *Stress, Stability and Chaos in Structural Engineering: An Energy Approach*. McGraw-Hill, London.
Hunt, G. W. (1971). *Perturbation Techniques in Elastic Stability*. PhD thesis. University College, London.
Huseyin, K. (1975). *Non-Linear Theory of Elastic Stability*. Noordhoff, Leyden, The Netherlands.
Koiter, W. T. (1945). *On the Stability of Elastic Equilibrium*, in Dutch, thesis, Delft, H. J. Paris, Amsterdam.
Nayfeh, A. H. (1981). *Introduction to Perturbation Techniques*. Wiley, New York.
Thompson, J. M. T. and Hunt, G. W. (1973). *A General Theory of Elastic Stability*. Wiley, London.